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On the Determination of a certain Class of Inequalities in the Moon's Motion. By Ernest W. Brown, B.A.

1. The two principal methods of treating the Lunar Theory—viz. (1) the *general* method, in which we obtain a general approximate solution, applying to any single Moon whose present orbit lies within given limits, in terms of certain constants, the coordinates of position and the time; and (2) the *special* method, in which we use the numerical values of those constants at the outset for any given Moon, and obtain a solution involving the coordinates of position and the time only—present each of them certain advantages. In the first case, we have the theory, worked out to a certain degree of accuracy, immediately applicable to any single Moon in our solar system, and therefore arranged in such a way that any small change which improved data may involve in the values of the constants can be made easily without requiring us to go over the whole of the work again. On the other hand, the number of terms which have been found necessary to secure a degree of accuracy commensurate with that of observation is very large, and it becomes a task of great labour to obtain them with any degree of certainty. In the second case, when we use numerical values from the start, the portions arising from the expansion of the functions in ascending powers of the constants involved, which portions in the first case would have to be neglected, are no longer left out of account. A great increase of accuracy naturally results. But this method is unsatisfactory in some respects. Having started with certain values of the constants for any particular Moon, we are bound to keep to those values; and also if any numerical mistake be made, it is not very easily traceable. The calculation is applicable only to the particular Moon selected. Any method, therefore, of finding the path of the Moon which will give at the same time numerical and algebraical results with equal accuracy would be of value.

Mr. G. W. Hill, in the *American Journal of Mathematics*, vol. i.,* has shown how to separate out those inequalities dependent *only* on the ratio of the mean motions of the Sun and Moon, and to obtain their coefficients both numerically and as an algebraic function in ascending powers of this ratio. He shows also that his method is not only susceptible of a very high degree of accuracy with comparatively little trouble, but that any higher degree of accuracy can be obtained without the necessity of going over a large part of the ground again. His method for this class of inequalities is as follows:—

2. Take as origin the centre of the Earth considered as a sphere, and refer the motion to rectangular axes revolving round the origin with the mean angular velocity n' of the Sun round the Earth. If we neglect all the differences from purely elliptic

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* *Researches in the Lunar Theory*, pp. 5, 129, 245

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motion, but those dependent on the mean motions of the Sun and Moon, the Sun will describe a circle about the Earth with constant angular velocity n', and we may therefore assume the moving axis of x to pass through the Sun. Under these circumstances, the equations of motions of the Moon become

$$\left.\begin{aligned} &\frac{d^2x}{dt^2}-2n'\frac{dy}{dt}+\left(\frac{\mu}{r^3}-3n'^2\right)x=0 \\ &\frac{d^2y}{dt^2}+2n'\frac{dx}{dt}+\frac{\mu}{r^3}y \qquad\quad =0 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{I.}$$

These equations will give us all inequalities from purely circular motion, which involve only the eccentricity of the Moon and the mean motions of the Sun and Moon. By taking certain "particular integrals" of these equations, the inequalities dependent only on the mean motions can be separated out and determined. This particular solution is given by

$$x=\Sigma a_{2i}\cos(2i+1)(n-n')(t-t_0)$$
$$y=\Sigma a_{2i}\sin(2i+1)(n-n')(t-t_0),$$

when i takes integral values positive and negative, and a_{2i} is a function only of the ratio of the mean motions of the order m^{2i} at least. The quantities a_{2i} are then determined by substituting this particular solution in the equations of motion. Certain transformations are first made in order to reduce the equations (I) to a convenient form.

3. In most previous methods, and especially in Delaunay's, the coefficients of this class of inequalities are expressed directly in ascending powers of $n'/n=m$; but in Mr. Hill's method of preparing the equations for solution the function of m which appears most naturally for use is $n'/(n-n')$. Let

$$m_1=\frac{n'}{n-n'},=\frac{m}{1-m};$$

we then have

$$m=\frac{m_1}{1+m_1}.$$

At first sight m_1 would seem to be a less suitable quantity than m, since m_1 is greater than m, and consequently a greater number of terms would seem to be required to obtain a given degree of accuracy. But it is shown that the numerical multipliers of the higher powers of m_1 are much smaller than those of the corresponding series in m; in other words, the series for the coefficients expressed in ascending powers of m_1 are so much more convergent than the same series expressed in powers of m, as to quite overbalance the disadvantage of m_1 being slightly greater than m. Further, he shows that by using a certain function of m_1 *obtained theoretically*, for expansion, the series can be ren-

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dered still more convergent. This function of m_1 is μ, where μ is given by

$$\mu = \frac{m_1}{1 - \frac{1}{3}m_1} = \frac{m}{1 - \frac{4}{3}m}.$$

Hence

$$m = \frac{\mu}{1 + \frac{4}{3}\mu}.$$

The series for a_{2i} are given by Mr. Hill, developed in terms of both m_1 and μ.

4. These transformations can be utilised also in another way. The inequalities found as indicated in § (2) are referred to rectangular coordinates. It has been usual to refer the Moon's motion to polar coordinates. We have for this purpose

$$r \cos v = \Sigma a_{2i} \cos 2i\,(n - n')\,(t - t_0)$$
$$r \sin v = \Sigma a_{2i} \sin 2i\,(n - n')\,(t - t_0),$$

where r is the radius vector and v the excess of the true over the mean longitude. From these r and v can be easily found.

The process of finding these coefficients as algebraic expansions in powers is somewhat laborious when, as in the special case of the Variation Inequality, it is necessary to carry the expansions to a high power. But there is no reason why, supposing Delaunay's expressions in longitude and parallax for these coefficients to be accurate as far as they go, we should not put in them

$$m = \frac{\mu}{1 + \frac{4}{3}\mu},$$

and expand in powers of μ. This transformation gives us smaller numerical multipliers of the higher powers, and consequently greater accuracy when we substitute for μ its numerical value. It is to be noted that if the given series is calculated as far as m^q, the transformed series cannot be carried further than μ^q.

5. As an example, take that part of Delaunay's expression* for the coefficient of the Variation which depends on m. It is

$$\begin{array}{lllll}
\frac{11}{8}m^2 & +\frac{59}{12}m^3 & +\frac{893}{72}m^4 & +\frac{2855}{108}m^5 & +\frac{8304449}{165888}m^6 \\
1586''\cdot8883 & +424''\cdot4474 & +80''\cdot0906 & +12''\cdot7689 & +1''\cdot8087 \\
+\frac{102859909}{1244160}m^7 & +\frac{7596606727}{74649600}m^8 & -\frac{8051418161}{1119744000}m^9 \;.\;\;.\;\;. & & (a) \\
+0''\cdot2234 & +0''\cdot0206 & -0''\cdot0001 & &
\end{array}$$

where the coefficients expressed in seconds are written underneath. Putting *in this series*

$$m = \mu/(1 + \tfrac{4}{3}\mu),$$

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* Delaunay, *Mémoires de l'Académie des Sciences*, tome xxix. p. 815.

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and expanding in powers of μ, it becomes

$$\begin{array}{ccccc}
\frac{11}{8}\mu^2 & +\frac{5}{4}\mu^3 & +\frac{5}{72}\mu^4 & -\frac{11}{36}\mu^5 & -\frac{82111}{165888}\mu^6 \\
1957''\cdot 9686 & +147''\cdot 8944 & +0''\cdot 6827 & -0''\cdot 2496 & -0''\cdot 0336 \\
\\
-\frac{350399}{138240}\mu^7 & -\frac{233559113}{74649600}\mu^8 & -\frac{10961275281}{1119744000}\mu^9 & \ldots & (\beta) \\
-0''\cdot 0143 & -0''\cdot 0015 & -0''\cdot 0004 & &
\end{array}$$

The values of the coefficients obtained from these expressions are—

$$\left.\begin{array}{lr}
\text{From } (\alpha) \quad \ldots \quad \ldots \quad \ldots & 2106''\cdot 2478 \\
\text{From } (\beta) \quad \ldots \quad \ldots \quad \ldots & 2106\cdot 2463 \\
\text{From Mr. Hill's values} \quad \ldots & 2106\cdot 2463
\end{array}\right\} \quad \ldots\ldots \quad (\gamma)$$

The last value has been calculated from the numerical values of the coefficients a_{2i} given in his paper already referred to. The equality of the second and third values—the second obtained from Delaunay's series when expressed in terms of μ, and the third from Mr. Hill's numerical values—is striking. It not only shows us the improvement in accuracy afforded by such transformations, but it gives us also some security for the general correctness of this particular part of Delaunay's expression for the coefficient of the variation.

6. There appears to be a kind of oscillation in the class of series of which (α) or (β) is an example. For instance, in the series (α) the numerical multipliers are positive, and increase gradually until a change of sign occurs between m^8 and m^9, when there is a sudden drop. The numerical values (γ) seem to show that there must be a large increase in the multipliers of the powers after m^9, and that they must have negative signs; so that probably the multipliers of m^{10} and m^{11} are much larger than even those of m^7 or m^8. The series (β) has a sudden drop between μ^3 and μ^4, and then a change of sign occurs, after which the multipliers gradually increase again. I have calculated similar expressions in the coefficients of other inequalities, and generally it appears that when we, by any such substitution as that made above, apparently improve the convergency of the series, the change of sign is brought nearer to the beginning of the series. An ideal to aim at would seem to be, firstly, that the numerical multipliers be made as small as possible; secondly, that there be no sudden increase in them in the later part of the series; and thirdly, that these two conditions should involve that there be no long run of powers with the same sign attached to them. This appears very forcibly in the expressions for the Parallactic Inequality given in § (11) below. It should be stated that as the series are calculated up to some definite power only,

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and as the law of progression of the series is not able to be expressed by an algebraical formula, owing to the complicated forms from which they arise, a substitution like that made above *must not be arbitrary*, but must be indicated by theory. An arbitrary substitution, which would seem to make the parts of the series already calculated converge rapidly, may make the unknown part very slowly convergent, and thus introduce unknown errors into the numerical values of the coefficients.

7. It has been mentioned above that the equations (I.) determine all the inequalities dependent on the eccentricity and the mean motion only. As in these equations when transformed for solution m_1 and not m is the quantity which appears, all those series which depend on the eccentricity and mean motions only will probably be rendered more convergent when expressed in terms of m_1. Consequently, in Delaunay's expressions involving e and m only, we are entitled to put $m=m_1/(1+m_1)$, and expand in powers of m_1 up to the same power of m_1 as that to which m was carried; the result of this should be more convergent series and greater accuracy in the coefficients when expressed in seconds. It will be found on examination that nearly all these series have a number of successive powers of m of the same sign with gradually increasing multipliers; and the transformation improves the series in the manner mentioned in § (6).

8. When we examine Delaunay's expression for the part of the coefficient of the Parallactic Inequality dependent on the ratios of the mean distances and mean motions, we notice that the numerical multipliers of the higher powers of m become very large. The multiplier of any one power is between four and seven times that of the previous power; and since m is in the case of our Moon about ·074, the series expressed in seconds converges very slowly. The expression is given below in § (11). In fact, the last term calculated, which is of the ninth order, corresponds to a coefficient of $0''\cdot38$; so that some method is necessary for this inequality which will give us either more terms easily, or a new form of expression with greater convergency. I shall give elsewhere the analysis by which the expressions and values below have been obtained, and expose only the method used to work up the coefficients into a more convergent form. No other inequalities but those given are affected thereby.

In obtaining these inequalities dependent on m and the ratio of the mean distances, some terms of the disturbing function have to be added to the left-hand sides of the equations (I.). The constant m_1 is still the function of m which appears naturally for use in expansion. The general method of procedure has been the same as that Mr. Hill used in getting the inequalities dependent on the mean motions only.

9. Let a denote the mean semi-axis major of the Moon's orbit, a' that of the Sun's orbit, v the difference between the true and mean longitudes of the Moon, 2D the argument of the variation

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in Delaunay's theory; the parts of the Lunar Inequalities dependent on the mean motions and a/a' only, appear in the following form:—

$$r\cos v = a_0 + (a_1 + a_{-1})\cos \mathrm{D} + (a_2 + a_{-2})\cos 2\mathrm{D} + \dots$$
$$r\sin v = \qquad (a_1 - a_{-1})\sin \mathrm{D} + (a_2 - a_{-2})\sin 2\mathrm{D} + \dots$$

The quantities $a_{\pm p}$ are functions of the kind,

$$\frac{a_{\pm 2p}}{a} = f(m_1) + \left(\frac{a}{a'}\right)^2 \phi(m_1) + \dots$$
$$\frac{a_{\pm 2p+1}}{a} = \frac{a}{a'}\mathrm{F}(m_1) + \left(\frac{a}{a'}\right)^3 \Phi(m_1) + \dots$$

where p is a positive integer.

In the functions f for a_{2p}, $m_1{}^{2p}$ is the lowest power of m_1 which occurs; in F the lowest power is also $m_1{}^{2p}$, except when $p=0$, when it is m_1; when $p=0$, f is of the form $1+\alpha m_1{}^2+\beta m_1{}^3+\,.\,.$, α, β, . . being numbers. Mr. Hill, in his paper referred to, has given the functions f, and his values for them have been used in obtaining the functions F, ϕ, Φ.

When we transform these expressions to polar coordinates, we have, neglecting cubes and higher powers of a/a',

$$v = \mathrm{A}\frac{a}{a'}\sin \mathrm{D} + \left\{\mathrm{B} + \mathrm{B}'\left(\frac{a}{a'}\right)\right\}\sin 2\mathrm{D} + \mathrm{C}\frac{a}{a'}\sin 3\mathrm{D} + \dots$$

where A, B, B′, C . . . are functions of m_1. D is the argument of the Parallactic Inequality, and the first term represents the principal part of it, *i.e.* the part dependent on m_1 and a/a' only. The full expression of this inequality contains terms dependent on the solar and lunar eccentricity and the lunar inclination, but we are not here concerned with such terms, and they do not affect the expressions which follow.

10. The principal part of the coefficient $\mathrm{A}a/a'$ depends chiefly on a_1/a and a_{-1}/a. In obtaining a_1 and a_{-1} as far as the order $m_1{}^5 .\ a/a'$, it is possible so to arrange the work that the equations determining them are linear and of the form

$$\frac{a_1}{a_0}\cdot\alpha_1 + \frac{a_{-1}}{a_0}\cdot\beta_1 = \kappa_1\cdot\frac{a}{a'},$$
$$\frac{a_1}{a_0}\cdot\alpha_2 + \frac{a_{-1}}{a_0}\cdot\beta_2 = -\kappa_2\cdot\frac{a}{a'},$$

where $\alpha_1\ \alpha_2\ \beta_1\ \beta_2\ \kappa_1\ \kappa_2$ are functions of m_1 expanded in ascending powers of m_1, and the signs in the equations are so arranged that these functions become positive numerical quantities in the special case of our Moon.

Solving these equations for a_1 and a_{-1} we have

$$\frac{a_1}{a_0} = -\frac{\kappa_1\beta_2 + \kappa_2\beta_1}{\beta_1\alpha_2 - \alpha_1\beta_2}\cdot\frac{a}{a'} \qquad \frac{a_{-1}}{a_0} = +\frac{\kappa_1\alpha_2 + \kappa_2\alpha_1}{\beta_1\alpha_2 - \alpha_1\beta_2}\cdot\frac{a}{a'}.$$

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The expressions on the left-hand sides of these equations, when for α_1 β_1, etc. are substituted their expansions in powers of m_1, are cumbrous. At the same time, when α_1 β_1, etc. are given their numerical values, we get a very close approximation to the results obtained by the more accurate method of using numerical values from the start. Each of these quantities, when expanded in powers of m_1, converges quickly and no very large numerical multipliers seem to appear. But the multipliers in all of them gradually increase, and consequently, when we express the products κ_1 β_2, κ_2 β_1, etc., in ascending powers of m_1, larger multipliers occur. It seems best, however, for simplicity to expand the numerators and denominators of the expressions for a_1/a_0 and a_{-1}/a_0, and leave the results in the form of fractions which have the same denominator. The loss in accuracy is very small. We have then

$$\frac{a_1}{a_0} = -\frac{\gamma(m_1)}{\tau}\,\frac{a}{a'} \qquad \frac{a_{-1}}{a_0} = +\frac{\delta(m_1)}{\tau}\cdot\frac{a}{a'}$$

where γ, δ, τ are functions of m_1 expanded in ascending powers.

11. The terms in longitude with arguments D, 3D, 5D, calculated in this way, become

$$\left[-\frac{\frac{15}{8}m_1+\frac{9}{4}m_1^2-\frac{1951}{128}m_1^3-\frac{41585}{1024}m_1^4-\frac{2096751}{49152}m_1^5}{1-4m_1-\frac{37}{8}m_1^2-\frac{17}{6}m_1^3-\frac{89963}{9216}m_1^4}\right.$$

$$\left.-\frac{35}{1024}m_1^4-\frac{1345}{12288}m_1^5\right]\times\frac{a}{a'}\sin D$$

$$-124''{\cdot}1493 \qquad -12''{\cdot}0447 \qquad +6''{\cdot}5970 \qquad +1''{\cdot}4218 \qquad +0''{\cdot}1207$$

$$-0''{\cdot}0008 \qquad -0''{\cdot}0002$$

$$\left[+\frac{15}{32}m_1^2+\frac{55}{128}m_1^3-\frac{41}{1536}m_1^4+\frac{2309}{9216}m_1^5-\right.$$

$$\left.\frac{\frac{255}{128}m_1^3+\frac{7543}{1536}m_1^4-\frac{134525}{8192}m_1^5}{1-4m_1-\frac{37}{8}m_1^2-\ldots}\right]\times\frac{a}{a'}\sin 3D$$

$$+1''{\cdot}6172 \qquad +0{\cdot}1199 \qquad -0''{\cdot}0006 \qquad +0''{\cdot}0005$$

$$-0''{\cdot}8624 \qquad -0''{\cdot}1718 \qquad +0''{\cdot}0470$$

$$\left[+\underbrace{\frac{75}{128}m_1^4+\frac{2797}{2048}m_1^5-\frac{18415}{8192}m_1^5\times\frac{1}{1-4m-\ldots}}_{+0''{\cdot}0093}\right]\times\frac{a}{a'}\sin 5D.$$

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For the sake of comparison I give Delaunay's expressions* for the same terms; they are expanded in powers of m.

$$-\left[\frac{15}{8}m+\frac{93}{8}m^2+\frac{6887}{128}m^3+\frac{137197}{512}m^4+\frac{4628333}{3072}m^5+\frac{63106813}{8192}m^6\right.$$

$$-74''{\cdot}0235 \quad -34''{\cdot}3297 \quad -11''{\cdot}8852 \quad -4''{\cdot}4276 \quad -1''{\cdot}8621 \quad -0''{\cdot}7122$$

$$\left.+\frac{10835537159}{196608}m^7\right]\frac{a}{a'}\sin \mathrm{D}$$

$$-0''{\cdot}3811$$

$$+\left[\frac{15}{32}m^2-\frac{5}{8}m^3-\frac{259}{16}m^4-\frac{666249}{6144}m^5\right]\frac{a}{a'}\sin 3\mathrm{D}$$

$$+1''{\cdot}3843 \quad -0''{\cdot}1381 \quad -0''{\cdot}2675 \quad -0''{\cdot}1340$$

$$+\left[\frac{75}{128}m^4+\frac{3911}{1024}m^5\right]\frac{a}{a'}\sin 5\mathrm{D}$$

$$+0''{\cdot}0097 \quad +0''{\cdot}0047$$

The gain in accuracy by the former mode of expression is evident; the terms there are carried only as far as $m_1{}^5$, while Delaunay's go as far as m^7.

The equations for obtaining a_1 a_{-1} involve largely a_2 a_{-2}, which go to form the coefficient of the Variation. It has been mentioned that the expansions of these quantities are much more convergent when expressed in terms of μ, § (3). Hence it is probable that the resulting expressions for the coefficients of the Parallactic Inequality would be also rendered more convergent when expressed in terms of this quantity. This inequality thus transformed becomes,

$$-\left[\frac{\frac{15}{8}\mu+\mu^2-\frac{2158}{128}\mu^3-\frac{175705}{9216}\mu^4+\frac{3261161}{442368}\mu^5}{1-\frac{13}{3}\mu-\frac{133}{72}\mu^2+\frac{91}{216}\mu^3-\frac{701891}{82944}\mu^4}+\frac{35}{1024}\mu^4+\frac{785}{12288}\mu^5\right]\frac{a}{a'}\sin \mathrm{D}.$$

$$-131''{\cdot}1311 \quad -5''{\cdot}8109 \quad +8''{\cdot}1438 \quad +0''{\cdot}7648 \quad -0''{\cdot}0246 \quad -0''{\cdot}0009 \quad -0''{\cdot}0001$$

[In making the transformation, both numerator and denominator of the expression given at the beginning of this section must be multiplied by m_1, and, in the result, for m_1 must be put $\mu/(1+\frac{1}{3}\mu)$. It is in that form that the expression occurs.] The series in both numerator and denominator seem slightly more convergent, and the numerical result is a little closer to the final value.

12. To exhibit the values of the coefficients obtained by using these several expressions, a table is given. In column (i.) is the argument of the term; in (ii.) the coefficients found by using the numerical value of m_1 from the start; (iii.) those found by

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* Delaunay, *Mém. cit.* pp. 847, 856, 860.

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putting the numerical values of α_1 β_1, etc. in the expressions of § (10); (iv.) gives the values from the expressions in terms of m_1 in § (ii.); (v.) those in terms of μ in the same section; in (vi.) are Delaunay's values. No estimate of neglected terms is taken into account.

i.	ii.	iii.	iv.	v.	vi.
D	−128″·069	−128″·073	−128″·056	−128″·059	−127′·621
3D	+ 0·750	...	+ 0·749	...	+ 0·845
5D	+ 0·008	...	+ 0·009	...	+ 0·014

The column (ii.) gives the final values to be used; (iii.), (iv.), or (v.) may serve also as checks on the numerical work. It may be noticed that in obtaining the numerical values from the outset it is only necessary to substitute for m_1 its value. The ratio a/a' occurs only in its first power; it may therefore be kept as a factor right through, and its numerical value substituted as the final step.

13. The expansions I have given for the parallactic terms, when transformed, agree with those of Delaunay as far as the terms of the sixth order. The numerical multipliers of $m^5.a/a'$ differ by small quantities; these differences would not, however, cause differences in the coefficients expressed in seconds of more than 0″·004. As the coefficient of the term with argument 5D is very easily obtainable by the methods indicated, I am inclined to attribute some small numerical errors to Delaunay. Newcomb suggests * that Delaunay's multiplier of $m^7 a/a'$ may be wrong. I have not yet been able to test Delaunay's multipliers of $m^6.a/a'$ and $m^7.a/a'$ owing to the large amount of labour they entail. But that the numerical values given in column (ii.) of the table do not differ by more than one or two thousandths of a second from the true values can be seen during the progress of the work necessary for finding those values.

14. The corresponding terms in the expression for the Moon's parallax, which I have obtained, are :—

$$-\frac{1}{a}\left[\frac{\frac{15}{16}m_1+\frac{3}{8}m_1{}^2-\frac{1869}{256}m_1{}^3}{1-4m-\frac{37}{8}m^2-\ \ldots}\right]\frac{a}{a'},\ \cos \mathrm{D}.$$

$$-1''{\cdot}0291 \quad -0''{\cdot}0333 \quad +0''{\cdot}0524 \quad =-1''{\cdot}0100.$$

$$+\frac{1}{a}\left[\frac{25}{64}m_1{}^2+\frac{65}{256}m_1{}^3-\frac{459}{256}\cdot\frac{m_1{}^3}{1-4m_1-\ \ldots}\right]\frac{a}{a'},\ \cos 3\mathrm{D}.$$

$$+0''{\cdot}0223 \quad +0''{\cdot}0012 \quad -0''{\cdot}0139 \quad =+0''{\cdot}0096.$$

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* Newcomb, *Transformation of Hansen's Lunar Theory*, Astronomical Papers for Use of American Ephemeris, vol. i., part ii.

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Delaunay's expressions * for the same being

$$-\frac{1}{a}\left[\frac{15}{16}m+\frac{81}{16}m^2+\frac{5817}{256}m^3\right]\frac{a}{a'}, \cos D.$$

$$-0''\cdot6136 \quad -0''\cdot2479 \quad -0''\cdot0832 \quad =-0''\cdot9447\cdot$$

$$+\frac{1}{a}\left[\frac{25}{64}m^2-\frac{115}{128}m^3\right]\frac{a}{a'}, \cos 3D.$$

$$+0''\cdot0191 \quad -0''\cdot0033 \quad =+0''\cdot0158\cdot$$

The coefficients obtained by using numerical values from the start are $-1''\cdot0010$ and $+0''\cdot0080$, to which the former pair approximate closely.

15. A comparison of these new values with those given by Hansen cannot be complete owing to the fact that Hansen's numerical values contain the terms due to eccentricity and inclination. But a rough estimate of the uncalculated parts of these series may be made from Delaunay's expressions, since they are carried to a higher degree of approximation for numerical purposes than the terms we have been considering. The whole coefficient of the Parallactic Inequality in the longitude then becomes

$$127''\cdot30$$

Newcomb's method of transformation † shows that this coefficient when expressed in Hansen's notation is

	124″·54
Hansen's coefficient is	124·43
Diff.	0·11

The difference is probably partly due to a wrong estimate of the omitted terms in Delaunay's coefficient.

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Secular Perturbations of the Earth's Orbit by Mars.

By R. T. A. Innes.

In Professor Hill's "Theory of *Jupiter* and *Saturn*" he mentions that the terms arising from the fifth powers of the eccentricities and inclination amount to one per cent. of those of the first order in the case of *Mars* and the Earth, and that a computation by Gauss's method would be desirable. I have therefore

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* Delaunay, *Mém. cit.* pp. 922, 923.

† Newcomb, *Mém. cit.*